Dynamical Equations in the ARPS and COAMPS Coordinate System

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Abstract

This "treatise" derives the dynamical equations in the popular coordinate transformations of non-hydrostatic numerical weather prediction models. Tensor jargon is kept to a minimum.

1 Introduction

The coordinate transformations used in the non-hydrostatic models ARPS (U. of Oklahoma) and COAMPS (U.S. Navy) are reviewed. The complete transformation involves two steps. The first step is a transformation to an orthogonal coordinate system wrapped on a sphere, with equal scale factors in the horizontal directions (a conformal coordinate system). The second is a transformation of the conformal system to a non-orthogonal, terrain-following system.

Will we begin the journey in the (x, y, z) Cartesian coordinate system. After a brief passage through cospherical coordinates (r, ψ, λ), we arrive at orthogonal, conformal coordinates (X, Y, Z). From there, we distort the vertical coordinate into a terrain-following coordinate in the (X, Y, ζ) system.

In order to keep the derivations uncluttered, no mention is made of rotating coordinate systems. The Coriolis force that would result from such an analysis is rather trivial to transform, since it involves no spatial derivatives. It is simpler just to graft the well-known Coriolis force into the final equations.

2 Some basics

The position of a particle in an inertial Cartesian reference frame can be written as:

$$\vec{r} = x_i \hat{\mathbf{i}}_i \tag{1}$$

Some of you who have been well-schooled in the world of tensors and coordinate transformations would welcome the use of superscripts on the coordinates, or $\vec{r} = x^i \hat{\mathbf{i}}_i$. Such a convention can be useful for non-orthogonal systems. In the latter part of this treatise we indeed consider non-orthogonal, terrain-following coordinate systems, but can do so quite effectively without using the high-powered tensor notation. Here we will stay with subscripted indices.

What do we mean by an inertial Cartesian reference frame? The distance between two points is given by:

$$(ds)^2 = dx_i dx_i, \tag{2}$$

and the acceleration that is related to fundamental forces is calculated by:

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2 x_i}{dt^2}\hat{\mathbf{i}}_i. \tag{3}$$

As you probably already know, non-Cartesian frames can be very beneficial if, for example, one of the coordinate curves is aligned with a force, like gravity. Hence the use of spherical coordinates in NWP, where the radial (or vertical) direction is aligned with gravity.

In general coordinates $q_i(x_1, x_2, x_3)$ the distance between two points is given by:

$$(ds)^2 = g_{ij} dq_i dq_j \tag{4}$$

where g_{ij} is the *fundamental metric tensor*. Coordinate systems are orthogonal if $g_{ij} = 0$ for $i \neq j$.

Let the *covariant base vectors* be given by

$$\mathbf{e}_i \equiv \frac{\partial \vec{r}}{\partial q_i} \tag{5}$$

or

$$\mathbf{e}_i \equiv \frac{\partial x_j}{\partial q_i}\hat{\mathbf{i}}_j. \tag{6}$$

Now,

$$d\vec{r} \cdot d\vec{r} = \frac{\partial \vec{r}}{\partial q_i} dq_i \cdot \frac{\partial \vec{r}}{\partial q_j} dq_j \tag{7}$$

so the fundamental metric tensor is given by

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j. \tag{8}$$

In an orthogonal coordinate system, only the three diagonal elements are nonzero with:

$$h_i^2 \equiv g_{ii} \text{ (no sum!)}. \tag{9}$$

The scale factors can be used to define unit base vectors:

$$\hat{\mathbf{e}}_i = \mathbf{e}_i / h_i \text{ (no sum!)}, \tag{10}$$

which gives the property that:

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}. \tag{11}$$

3 Spherical Coordinates

Rather than use numerical subscripts for the coordinates, we can also use unsubscripted variables. For example, the position vector can be written as:

$$\vec{r} = x\widehat{\mathbf{\imath}} + y\widehat{\mathbf{\jmath}} + z\widehat{\mathbf{k}}. \tag{12}$$

Spherical coordinates can be defined in terms of a radius r, a co-latitude ψ and a longitude λ. These coordinates can be related to Cartesian coordinates (x,y,z) by:

$$\begin{aligned} x &= r\cos\lambda\sin\psi \\ y &= r\sin\lambda\sin\psi \\ z &= r\cos\psi. \end{aligned} \tag{13}$$

and

$$\begin{aligned} \lambda &= \arctan(x, y) \\ \psi &= \arctan(z, \sqrt{x^2 + y^2}) \\ r &= \sqrt{x^2 + y^2 + z^2}. \end{aligned} \tag{14}$$

The covariant base vectors are:

$$\begin{aligned} \mathbf{e}_r &= \cos\lambda\sin\psi\widehat{\mathbf{\imath}} + \sin\lambda\sin\psi\widehat{\mathbf{\jmath}} + \cos\psi\widehat{\mathbf{k}} \\ \mathbf{e}_\lambda &= -r\sin\lambda\sin\psi\widehat{\mathbf{\imath}} + r\cos\lambda\sin\psi\widehat{\mathbf{\jmath}} \\ \mathbf{e}_\psi &= r\cos\lambda\cos\psi\widehat{\mathbf{\imath}} + r\sin\lambda\cos\psi\widehat{\mathbf{\jmath}} - r\sin\psi\widehat{\mathbf{k}}. \end{aligned} \tag{15}$$

It is easy to show that the spherical coordinates are orthogonal with $\mathbf{e}_r \cdot \mathbf{e}_\lambda = 0$, $\mathbf{e}_r \cdot \mathbf{e}_\psi = 0$ and $\mathbf{e}_\lambda \cdot \mathbf{e}_\psi = 0$. The scale factors are also easily found: $h_r = 1$, $h_\psi = r$ and $h_\lambda = r\sin\psi$.

4 Lambert Conformal Coordinates

Consider coordinates (X, Y, Z) related to spherical coordinates by

$$\begin{aligned} \lambda &= \lambda(X, Y) \\ \psi &= \psi(X, Y) \\ r &= Z + a, \end{aligned} \tag{16}$$

where a is the radius of the Earth. How can we chose $\lambda(X, Y)$ and $\psi(X, Y)$ to make an orthogonal coordinate system?

$$\begin{aligned} \mathbf{e}_X &= \frac{\partial\vec{r}}{\partial X} \\ &= \frac{\partial\vec{r}}{\partial\lambda}\frac{\partial\lambda}{\partial X} + \frac{\partial\vec{r}}{\partial\psi}\frac{\partial\psi}{\partial X} + \frac{\partial\vec{r}}{\partial r}\frac{\partial r}{\partial X} \\ &= \mathbf{e}_\lambda\frac{\partial\lambda}{\partial X} + \mathbf{e}_\psi\frac{\partial\psi}{\partial X}. \end{aligned} \tag{17}$$

Similarly,

$$\mathbf{e}_Y = \mathbf{e}_\lambda \frac{\partial \lambda}{\partial Y} + \mathbf{e}_\psi \frac{\partial \psi}{\partial Y} \tag{18}$$

and

$$\mathbf{e}_Z = \mathbf{e}_r. \tag{19}$$

The coordinates will be orthogonal if $\mathbf{e}_X \cdot \mathbf{e}_Y = 0$, or if

$$\mathbf{e}_\lambda \cdot \mathbf{e}_\lambda \frac{\partial \lambda}{\partial X}\frac{\partial \lambda}{\partial Y} + \mathbf{e}_\psi \cdot \mathbf{e}_\psi \frac{\partial \psi}{\partial X}\frac{\partial \psi}{\partial Y} = 0 \tag{20}$$

Here is a clever transformation that somebody (M. Lambert?) has come up with. It's related to making maps by projecting the surface of a sphere into a cone, cutting the cone, and squashing it into a plane. We will experience the joy of at least partially deriving the transformation. For the purpose of deriving the transformation and the dynamical equations, we really don't need to mention the historical roots of this transformation in the map making process, meaning the projecting and squashing of cones are not mentioned further.

Let

$$Q \equiv \sqrt{X^2 + Y^2} \tag{21}$$

and restrict our search for ψ and λ to

$$\psi(X, Y) = \psi(Q) \tag{22}$$

and

$$\lambda(X, Y) = \frac{1}{n}\arctan(-Y, X) + \lambda_0. \tag{23}$$

Thus the $-Y$ axis goes down the longitude λ_0. (Note: $\arctan(-Y, X)$ preserves information about the quadrant in the calculation of arctan. It is the same as $\arctan(\frac{X}{-Y})$ when both the numerator and denominator are positive.)

Our only task in this "derivation" is to find $\psi(Q)$. Using (22) and (23) in (20) we have

$$r^2 \sin^2 \psi \frac{\partial \lambda}{\partial X}\frac{\partial \lambda}{\partial Y} + r^2 \frac{\partial \psi}{\partial X}\frac{\partial \psi}{\partial Y} = 0 \tag{24}$$

or

$$r^2 \sin^2 \psi \frac{\partial \lambda}{\partial X}\frac{\partial \lambda}{\partial Y} + r^2 \frac{\partial Q}{\partial X}\frac{\partial Q}{\partial Y}\left(\frac{d\psi}{dQ}\right)^2 = 0. \tag{25}$$

Now with

$$\frac{\partial}{\partial X}\arctan(-Y, X) = -\frac{Y}{X^2 + Y^2}, \tag{26}$$

$$\frac{\partial}{\partial Y}\arctan(-Y, X) = \frac{X}{X^2 + Y^2}, \tag{27}$$

$$\frac{\partial Q}{\partial X} = \frac{X}{\sqrt{X^2 + Y^2}}, \tag{28}$$

and

$$\frac{\partial Q}{\partial Y} = \frac{Y}{\sqrt{X^2 + Y^2}}, \tag{29}$$

(25) becomes

$$-\frac{XY}{n^2(X^2+Y^2)^2}\sin^2\psi + \frac{XY}{X^2+Y^2}\left(\frac{d\psi}{dQ}\right)^2 = 0. \tag{30}$$

This simplifies to

$$\frac{d\psi}{dQ} = \frac{\sin\psi}{nQ}, \tag{31}$$

or

$$\frac{d\psi}{\sin\psi} = \frac{dQ}{nQ}. \tag{32}$$

A solution is

$$\ln\tan\left(\frac{\psi}{2}\right) = \frac{1}{n}\ln Q + C \tag{33}$$

or

$$\psi = 2\arctan\left[\left(\frac{Q}{b}\right)^{\frac{1}{n}}\right] \tag{34}$$

where now b is the integration constant to be determined. We also will have use for the inverse:

$$Q = b\tan^n\left(\frac{\psi}{2}\right). \tag{35}$$

What are the best values to use for b and n? To answer this question, we first find h_X and h_Y. With $h_X^2 = \mathbf{e}_X \cdot \mathbf{e}_X$ and $h_Y^2 = \mathbf{e}_Y \cdot \mathbf{e}_Y$ we have from (17) that:

$$\begin{aligned} h_X^2 &= \mathbf{e}_\lambda \cdot \mathbf{e}_\lambda \frac{\partial\lambda}{\partial X}\frac{\partial\lambda}{\partial X} + \mathbf{e}_\psi \cdot \mathbf{e}_\psi \frac{\partial\psi}{\partial X}\frac{\partial\psi}{\partial X} \\ &= r^2\sin^2\psi\left(\frac{\partial\lambda}{\partial X}\right)^2 + r^2\left(\frac{d\psi}{dQ}\frac{\partial Q}{\partial X}\right)^2 \\ &= r^2\sin^2\psi\left(\frac{Y}{n(X^2+Y^2)}\right)^2 + r^2\sin^2\psi\left(\frac{X}{n(X^2+Y^2)}\right)^2 \\ &= \left(\frac{r\sin\psi}{nQ}\right)^2. \end{aligned} \tag{36}$$

Similarly for h_Y^2. So we find from (17) and (18)

$$h_X = \frac{r\sin\psi}{nQ}, \tag{37}$$

$$h_Y = \frac{r\sin\psi}{nQ}. \tag{38}$$

We will have use for the so-called *map factor*:

$$m \equiv \frac{1}{h_X} = \frac{1}{h_Y}. \tag{39}$$

The map factor is the ratio of distance traveled in the coordinate system to the physical distance traveled on the Earth's surface. (The fact that the map factor is isotropic in the horizontal earns this transformation the title of *conformal*. When a map is plotted out on a piece of paper with a Cartesian representation for X and Y, angles on the map will be the same as angles on the globe. For NWP, this fact is probably of little consequence). Here we have found

$$m = \frac{nb \tan^n \left(\frac{\psi}{2}\right)}{r \sin \psi} \tag{40}$$

It is sometimes useful to have $m = 1$ at a certain co-latitude ψ_t (the "true" latitude) and at $r = a$ where a is the radius of the Earth. Setting $h_X = 1$ in (37) and using (35) we have

$$b = a \frac{\sin \psi_t}{n \tan^n \left(\psi_t/2\right)} \tag{41}$$

What advantage is there to having $m \approx 1$ in the area of interest? Certainly if you forget to include m in one of your equations or subroutines, your error will be reduced!

What value should we use for n? It is often advantageous to have m nearly uniform across an area of interest. We may want to have m assume an extreme value at a certain colatitude ψ_m. We solve for $\psi = \psi_m$ in

$$\frac{\partial m}{\partial \psi} = \frac{nb}{r} \frac{\partial}{\partial \psi} \frac{\tan^n \left(\frac{\psi}{2}\right)}{\sin \psi} = 0 \tag{42}$$

Using **Mathematica** I found that

$$\frac{\partial}{\partial \psi} \frac{\tan^n \left(\frac{\psi}{2}\right)}{\sin \psi} = (n - \cos(\psi)) \csc^2(\psi) \tan^n(\frac{\psi}{2}) \tag{43}$$

Thus to have $\frac{\partial m}{\partial \psi} = 0$ at ψ_m we use $n = \cos \psi_m$. A special case of particular interest and simplicity is $\psi_m = 0$ or $n = 1$. This gives a *polar stereographic projection*. The Lambert conformal coordinate system may also be configured by specifying two colatitudes, ψ_1 and ψ_2, where $m = 1$. From (40), merely setting m equal at the two latitudes allows a solution for n in:

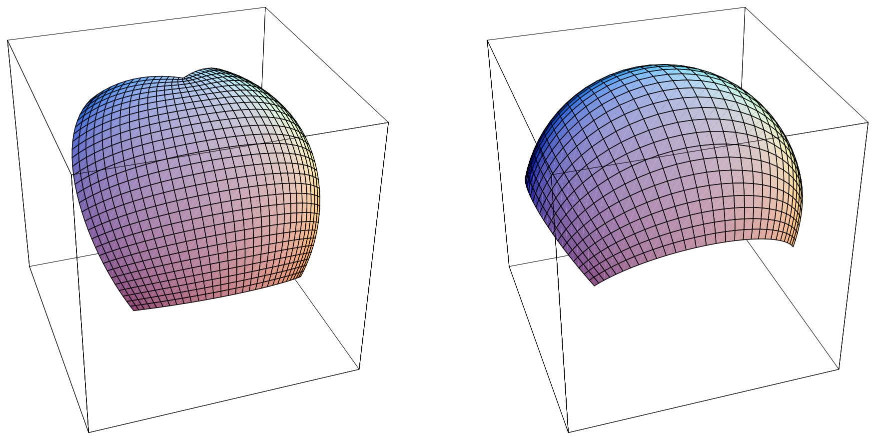
$$\frac{\tan^n(\psi_1/2)}{\sin(\psi_1)} = \frac{\tan^n(\psi_2/2)}{\sin(\psi_2)} \ . \tag{44}$$

Let

$$\alpha \equiv \frac{\sin(\psi_1)}{\sin(\psi_2)} \ . \tag{45}$$

and

$$\beta \equiv \frac{\tan(\psi_1/2)}{\tan(\psi_2/2)} \ . \tag{46}$$



Lambert, $\psi_t = \psi_m = .8$

Polar stereographic, $\psi_t = \psi_m = 0$

Figure 1: Some conformal grids on a sphere.

Lambert, $\psi_t = \psi_m = .8$:

Polar stereographic, $\psi_t = \psi_m = 0$:

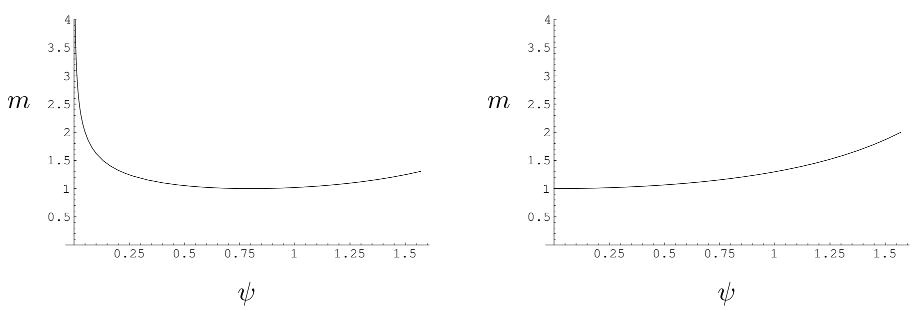


Figure 2: Map factors as a function of colatitude ψ for the grids in Fig. 1.

The solution for n is

$$n = \frac{\ln(\alpha)}{\ln(\beta)} \; . \tag{47}$$

The solution for b can be obtained from (40) by setting $m = 1$ and using the solution for n, as in (41) (with ψ_t replaced by ψ_1 or ψ_2).

5 The dynamical equations with map factors

The following formulas are useful for expressing the dynamical creatures in our forecast equations in terms of orthogonal curvilinear coordinates. The "proofs" I offer are mainly "advertisements" inviting you to "invest" in the formulas.

With

$$d\vec{r} = h_1 dq_1 \hat{\mathbf{e}}_1 + h_2 dq_2 \hat{\mathbf{e}}_2 + h_3 dq_3 \hat{\mathbf{e}}_3, \tag{48}$$

and with the desire for ∇A to have the property that

$$dA = \nabla A \cdot d\vec{r}, \tag{49}$$

then surely we must have

$$\nabla A = \frac{1}{h_1}\frac{\partial A}{\partial q_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial A}{\partial q_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial A}{\partial q_3}\hat{\mathbf{e}}_3. \tag{50}$$

Now $\nabla \cdot \vec{A}$ has the property that it can be used in Gauss's theorem:

$$\int_V \nabla \cdot \vec{A} dV = \int_S \vec{A} \cdot \widehat{\mathbf{n}} dS \tag{51}$$

Therefore,

$$\nabla \cdot \vec{A} = \lim_{V \to 0} \frac{1}{V} \int_S \vec{A} \cdot \widehat{\mathbf{n}} dS \tag{52}$$

Thus $\nabla \cdot \vec{A}$ can be derived by considering a little *cuboid* of the coordinate system. What is a "cuboid"? Actually, I just made up the word. For a very small cuboid, the volume is $V = h_1 h_2 h_3 dq_1 dq_2 dq_3$. Also, in a surface integral over a very small cuboid, $\vec{A} \cdot \widehat{\mathbf{n}}$ can be assumed to be constant in a face of the cuboid. A consideration of the surface integral over a cuboid gives:

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial h_2 h_3 A_1}{\partial q_1} + \frac{\partial h_1 h_3 A_2}{\partial q_2} + \frac{\partial h_1 h_2 A_3}{\partial q_3}\right) \tag{53}$$

Now $\nabla \times \vec{A}$ has the property that it can be used in Stokes's theorem:

$$\int_S \nabla \times \vec{A} \cdot \widehat{\mathbf{n}} dS = \oint \vec{A} \cdot d\vec{r} \tag{54}$$

Therefore,

$$\nabla \times \vec{A} \cdot \widehat{\mathbf{n}} = \lim_{S \to 0} \frac{1}{S} \oint \vec{A} \cdot d\vec{r} \tag{55}$$

By examining Stokes's theorem on the faces of very small cuboids we have:

$$\nabla\times\vec{A} = \frac{1}{h_2h_3}\left(\frac{\partial h_3A_3}{\partial q_2}-\frac{\partial h_2A_2}{\partial q_3}\right)\hat{\mathbf{e}}_1+\frac{1}{h_1h_3}\left(\frac{\partial h_1A_1}{\partial q_3}-\frac{\partial h_3A_3}{\partial q_1}\right)\hat{\mathbf{e}}_2+\frac{1}{h_1h_2}\left(\frac{\partial h_2A_2}{\partial q_1}-\frac{\partial h_1A_1}{\partial q_2}\right)\hat{\mathbf{e}}_3 \tag{56}$$

In transforming the equations to a non-Cartesian coordinate system, the term $\vec{U}\cdot\nabla\vec{U}$ can be quite troublesome. Using the above identities, we can derive the orthogonal curvilinear form for $\vec{U}\cdot\nabla\vec{U}$ using

$$\vec{U}\cdot\nabla\vec{U} = -\vec{U}\times\left(\nabla\times\vec{U}\right)+\nabla\frac{\vec{U}\cdot\vec{U}}{2} \tag{57}$$

The chore of this calculation is greatly relieved by the use of **Mathematica** or some other computer-based symbolic math processor. The result is:

$$\vec{U}\cdot\nabla\vec{U} = \sum_i\sum_j\frac{U_j}{h_j}\left(\frac{\partial U_i}{\partial x_j}+\frac{U_i}{h_i}\frac{\partial h_i}{\partial x_j}-\frac{U_j}{h_i}\frac{\partial h_j}{\partial x_i}\right)\hat{\mathbf{e}}_i. \tag{58}$$

With $\vec{U}=(U,V,W)$ in the (X,Y,Z) system, and with $h_Z=1$, we have

$$\begin{aligned}\vec{U}\cdot\nabla\vec{U} \;=\; & \\ & \left(\frac{U}{h_X}\frac{\partial U}{\partial X}+\frac{V}{h_Y}\frac{\partial U}{\partial Y}+W\frac{\partial U}{\partial Z}+\frac{UV}{h_Xh_Y}\frac{\partial h_X}{\partial Y}-\frac{V^2}{h_Yh_X}\frac{\partial h_Y}{\partial X}+\frac{UW}{h_X}\frac{\partial h_X}{\partial Z}\right)\hat{\mathbf{e}}_X\\ + & \left(\frac{U}{h_X}\frac{\partial V}{\partial X}+\frac{V}{h_Y}\frac{\partial V}{\partial Y}+W\frac{\partial V}{\partial Z}+\frac{UV}{h_Xh_Y}\frac{\partial h_Y}{\partial X}-\frac{U^2}{h_Yh_X}\frac{\partial h_X}{\partial Y}+\frac{VW}{h_Y}\frac{\partial h_Y}{\partial Z}\right)\hat{\mathbf{e}}_Y\\ + & \left(\frac{U}{h_X}\frac{\partial W}{\partial X}+\frac{V}{h_Y}\frac{\partial W}{\partial Y}+W\frac{\partial W}{\partial Z}-\frac{U^2}{h_X}\frac{\partial h_X}{\partial Z}-\frac{V^2}{h_Y}\frac{\partial h_Y}{\partial Z}\right)\hat{\mathbf{e}}_Z.\end{aligned} \tag{59}$$

Now with $h_X=\frac{1}{m}$, $h_Y=\frac{1}{m}$, neglecting curvature terms with W, and neglecting the curvature terms in the vertical direction:

$$\begin{aligned}\vec{U}\cdot\nabla\vec{U} \;=\; & \\ & \left(mU\frac{\partial U}{\partial X}+mV\frac{\partial U}{\partial Y}+W\frac{\partial U}{\partial Z}-UV\frac{\partial m}{\partial Y}+V^2\frac{\partial m}{\partial X}\right)\hat{\mathbf{e}}_X\\ + & \left(mU\frac{\partial V}{\partial X}+mV\frac{\partial V}{\partial Y}+W\frac{\partial V}{\partial Z}+U^2\frac{\partial m}{\partial Y}-UV\frac{\partial m}{\partial X}\right)\hat{\mathbf{e}}_Y\\ + & \left(mU\frac{\partial W}{\partial X}+mV\frac{\partial W}{\partial Y}+W\frac{\partial W}{\partial Z}\right)\hat{\mathbf{e}}_Z.\end{aligned} \tag{60}$$

One advantage to using $\frac{\partial m}{\partial X}\approx 0$ is that if you forget to put the curvature terms in your equations of motion, the error will be small! The other advantage is that, with m nearly uniform across the domain, a grid uniform in ΔX will be uniform in physical space also.

A "bare-bones" system of equations – with a pressure gradient force, Coriolis force, and gravity – is given by:

$$
\begin{aligned}
&\frac{\partial U}{\partial t} + mU\frac{\partial U}{\partial X} + mV\frac{\partial U}{\partial Y} + W\frac{\partial U}{\partial Z} - UV\frac{\partial m}{\partial Y} + V^2\frac{\partial m}{\partial X} + \frac{m}{\rho}\frac{\partial p}{\partial x} - fV = 0 \\
&\frac{\partial V}{\partial t} + mU\frac{\partial V}{\partial X} + mV\frac{\partial V}{\partial Y} + W\frac{\partial V}{\partial Z} + U^2\frac{\partial m}{\partial Y} - UV\frac{\partial m}{\partial X} + \frac{m}{\rho}\frac{\partial p}{\partial y} + fU = 0 \\
&\frac{\partial W}{\partial t} + mU\frac{\partial W}{\partial X} + mV\frac{\partial W}{\partial Y} + W\frac{\partial W}{\partial Z} + \frac{1}{\rho}\frac{\partial p}{\partial Z} - g = 0
\end{aligned} \tag{61}
$$

The fifth and sixth terms of in the first two equations of (61) are neglected in COAMPS. COAMPS is usually run with $\frac{\partial m}{\partial Y} = 0$ and $\frac{\partial m}{\partial X} = 0$ near the center of the domain, and presumably the terms also stay negligible throughout a small domain. Another way of stating why this is so, is that the coordinate lines are nearly great circles for small domains. For larger domains (see the example in Figure 1), there could be problems away from the center of the domain.

If mass is conserved in a model then we have:

$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0 \tag{62}
$$

and we can write:

$$
\rho\frac{\partial U}{\partial t} + \rho\vec{U}\cdot\nabla U = \frac{\partial \rho U}{\partial t} + \nabla\cdot(\rho\vec{U}U) \tag{63}
$$

Using (53) with the map factors we have:

$$
\nabla\cdot(\rho\vec{U}U) = m^2\frac{\partial}{\partial X}\frac{\rho UU}{m} + m^2\frac{\partial}{\partial Y}\frac{\rho VU}{m} + \frac{\partial}{\partial Z}(\rho WU). \tag{64}
$$

Some Canadian NWP models have made use of the following form of the equations. With:

$$
(U, V, W) \equiv \left(m\tilde{U}, m\tilde{V}, W\right), \tag{65}
$$

the bare bones system (61) becomes

$$
\begin{aligned}
&\frac{\partial \tilde{U}}{\partial t} + m^2\tilde{U}\frac{\partial \tilde{U}}{\partial X} + m^2\tilde{V}\frac{\partial \tilde{U}}{\partial Y} + W\frac{\partial \tilde{U}}{\partial Z} + \frac{\tilde{U}^2 + \tilde{V}^2}{2}\frac{\partial m^2}{\partial X} + \frac{1}{\rho}\frac{\partial p}{\partial x} - f\tilde{V} = 0 \\
&\frac{\partial \tilde{U}}{\partial t} + m^2\tilde{U}\frac{\partial \tilde{V}}{\partial X} + m^2\tilde{V}\frac{\partial \tilde{V}}{\partial Y} + W\frac{\partial \tilde{V}}{\partial Z} + \frac{\tilde{U}^2 + \tilde{V}^2}{2}\frac{\partial m^2}{\partial Y} + \frac{1}{\rho}\frac{\partial p}{\partial y} + f\tilde{U} = 0 \\
&\frac{\partial W}{\partial t} + m^2\tilde{U}\frac{\partial W}{\partial X} + m^2\tilde{V}\frac{\partial W}{\partial Y} + W\frac{\partial W}{\partial Z} + \frac{1}{\rho}\frac{\partial p}{\partial Z} - g = 0.
\end{aligned} \tag{66}
$$

5.1 A question I have

The Lambert conformal transformation is more complicated than the polar stereographic projection and has strange pathalogies near the pole. Why not just tilt a polar stereographic projection so that it's pole is over ψ_0 and λ_0? I don't know why that's not done in NWP...

6 Terrain-following coordinates

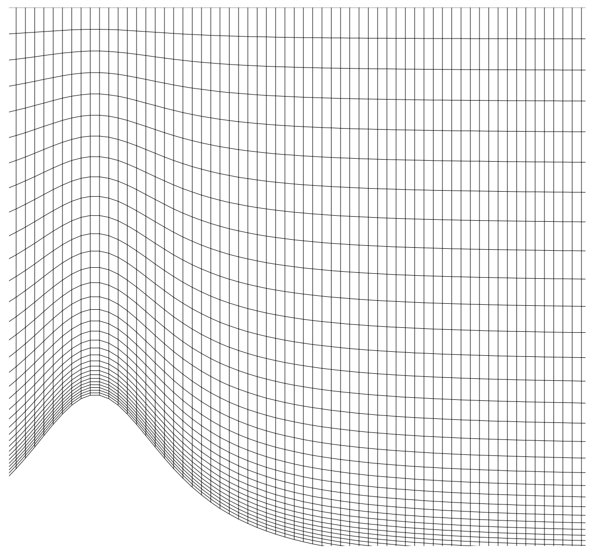


Figure 3: Vertical slice of a terrain-following coordinate system.

In numerical weather prediction, we store the values of variables on a grid, with the lowest grid point at the Earth's surface. The construction of the numerical model is simplified if a vertical coordinate $\zeta(X,Y,Z)$ is constructed that has a constant value of ζ at the surface (usually $\zeta = 0$). Above the Earth's surface, the surfaces of constant ζ will tend be displaced upwards by the terrain underneath. In conformal coordinates, the gradient of a scalar A is given by:

$$\nabla A = m \left(\frac{\partial A}{\partial X}\right)_{Y,Z} \hat{\mathbf{e}}_X + m \left(\frac{\partial A}{\partial Y}\right)_{X,Z} \hat{\mathbf{e}}_Y + \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \hat{\mathbf{e}}_Z. \tag{67}$$

The subscripts on the partial derivatives indicate which coordinates are being held constant. It would be cumbersome to make a numerical approximation to a derivative such as $\left(\frac{\partial A}{\partial X}\right)_{Y,Z}$, because neighboring values of A are not available on a constant level of Z. So we prefer to find forms for ∇A that use partial derivatives with ζ held constant, rather than Z.

We do not have to revisit the troublesome curvature terms associated with $\vec{U} \cdot \nabla \vec{U}$ because we will leave the component directions for $\vec{U}$ as they were in the conformal coordinate system. Both U and V will remain exactly the horizontal components, not the terrain following components. Thus, in transforming the acceleration term $\vec{U} \cdot \nabla \vec{U}$ to the terrain-following coordinate system, we will only deal with terms like $\vec{U} \cdot \nabla U$ with U treated as a scalar. Likewise, we leave the unit vectors in the gradient to be those of the conformal system. We are only making a "partial" conversion to the terrain-following system (pun intended!).

6.1 Converting partial derivatives to the terrain-following system

In order to convert creatures like $\vec{U} \cdot \nabla A$ to the terrain-following system, we return to the definition of the differential:

$$dA = \left(\frac{\partial A}{\partial X}\right)_{Y,Z} dX + \left(\frac{\partial A}{\partial Y}\right)_{X,Z} dY + \left(\frac{\partial A}{\partial Z}\right)_{X,Y} dZ. \tag{68}$$

If we hold Y constant and move on a constant ζ surface:

$$dA = \left(\frac{\partial A}{\partial X}\right)_{Y,Z} dX + \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \left(\frac{\partial Z}{\partial X}\right)_{Y,\zeta} dX, \tag{69}$$

which can be arranged as:

$$\left(\frac{\partial A}{\partial X}\right)_{Y,\zeta} = \left(\frac{\partial A}{\partial X}\right)_{Y,Z} + \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \left(\frac{\partial Z}{\partial X}\right)_{Y,\zeta}. \tag{70}$$

We do the same with holding X constant and moving on a constant ζ surface:

$$dA = \left(\frac{\partial A}{\partial Y}\right)_{X,Z} dX + \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \left(\frac{\partial Z}{\partial Y}\right)_{Y,\zeta} dY, \tag{71}$$

yielding:

$$\left(\frac{\partial A}{\partial Y}\right)_{X,\zeta} = \left(\frac{\partial A}{\partial Y}\right)_{X,Z} + \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \left(\frac{\partial Z}{\partial Y}\right)_{X,\zeta}. \tag{72}$$

Derivatives in the vertical direction convert as:

$$dA = \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \left(\frac{\partial Z}{\partial \zeta}\right)_{X,Y} d\zeta \tag{73}$$

or

$$\left(\frac{\partial A}{\partial \zeta}\right)_{X,Y} = \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \left(\frac{\partial Z}{\partial \zeta}\right)_{X,Y}. \tag{74}$$

So (67) converts to the terrain-following system as:

$$\begin{aligned} \nabla A &= \left[\left(m\frac{\partial A}{\partial X}\right)_{Y,\zeta} - \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \left(m\frac{\partial Z}{\partial X}\right)_{Y,\zeta}\right] \mathbf{\hat{e}}_X \\ &+ \left[\left(m\frac{\partial A}{\partial Y}\right)_{X,\zeta} - \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \left(m\frac{\partial Z}{\partial Y}\right)_{Y,\zeta}\right] \mathbf{\hat{e}}_Y + \left(\frac{\partial A}{\partial Z}\right)_{X,Y} \mathbf{\hat{e}}_Z. \end{aligned} \tag{75}$$

We can make this a bit neater by noticing that all partial derivatives are within the terrain-following coordinates, meaning ζ is held constant, rather than Z. From here on, we will drop the explicit designation of which coordinates are being held constant when all the derivatives are in the terrain-following coordinates, so

$$\nabla A = \left(m\frac{\partial A}{\partial X} - \frac{\partial A}{\partial Z} m \frac{\partial Z}{\partial X}\right) \mathbf{\hat{e}}_X + \left(m\frac{\partial A}{\partial Y} - \frac{\partial A}{\partial Z} m \frac{\partial Z}{\partial Y}\right) \mathbf{\hat{e}}_Y + \frac{\partial A}{\partial Z}\mathbf{\hat{e}}_Z. \tag{76}$$

6.2 $\vec{U} \cdot \nabla A$ in the terrain-following system

Recall that in the conformal system:

$$\vec{U} \cdot \nabla A = Um \left(\frac{\partial A}{\partial X} \right)_{Y,Z} + Vm \left(\frac{\partial A}{\partial Y} \right)_{X,Z} + W \left(\frac{\partial A}{\partial Z} \right)_{X,Y}. \tag{77}$$

Using (76), we have in the terrain-following system:

$$\vec{U} \cdot \nabla A = Um \left[\frac{\partial A}{\partial X} - \frac{\partial A}{\partial Z} \frac{\partial Z}{\partial X} \right] + Vm \left[\frac{\partial A}{\partial Y} - \frac{\partial A}{\partial Z} \frac{\partial Z}{\partial Y} \right] + W \frac{\partial A}{\partial Z} \tag{78}$$

or

$$\vec{U} \cdot \nabla A = Um \frac{\partial A}{\partial X} + Vm \frac{\partial A}{\partial Y} + \left(-Um \frac{\partial Z}{\partial X} - Vm \frac{\partial Z}{\partial Y} + W \right) \frac{\partial A}{\partial Z}. \tag{79}$$

We define a new vertical velocity (a *contravariant* vertical velocity, in tensor jargon):

$$\mathcal{W} \equiv \frac{-Um \frac{\partial Z}{\partial X} - Vm \frac{\partial Z}{\partial Y} + W}{\frac{\partial Z}{\partial \zeta}}, \tag{80}$$

which leaves the advection of A rather neat:

$$\vec{U} \cdot \nabla A = Um \frac{\partial A}{\partial X} + Vm \frac{\partial A}{\partial Y} + \mathcal{W} \frac{\partial A}{\partial \zeta}. \tag{81}$$

6.3 A "conservative" form for ∇A

If we multiply (76) by $\frac{\partial \zeta}{\partial Z} \frac{\partial Z}{\partial \zeta}$, we obtain another useful form of ∇A :

$$\nabla A = \frac{\partial \zeta}{\partial Z} \left\{ \left[\frac{\partial Z}{\partial \zeta} m \frac{\partial A}{\partial X} - \frac{\partial A}{\partial \zeta} m \frac{\partial Z}{\partial X} \right] \hat{\mathbf{e}}_X + \left[\frac{\partial Z}{\partial \zeta} m \frac{\partial A}{\partial Y} - \frac{\partial A}{\partial \zeta} m \frac{\partial Z}{\partial Y} \right] \hat{\mathbf{e}}_Y + \frac{\partial A}{\partial \zeta} \hat{\mathbf{e}}_Z \right\} \tag{82}$$

or

$$\nabla A = \frac{\partial \zeta}{\partial Z} \left\{ \left[m \frac{\partial}{\partial X} \left(A \frac{\partial Z}{\partial \zeta} \right) - \frac{\partial}{\partial \zeta} \left(Am \frac{\partial Z}{\partial X} \right) \right] \hat{\mathbf{e}}_X + \left[m \frac{\partial}{\partial Y} \left(A \frac{\partial Z}{\partial \zeta} \right) - \frac{\partial}{\partial \zeta} \left(Am \frac{\partial Z}{\partial Y} \right) \right] \hat{\mathbf{e}}_Y + \frac{\partial A}{\partial \zeta} \hat{\mathbf{e}}_Z \right\}. \tag{83}$$

This last form is close to what is called the *conservative form* of ∇A. Its use in finite-difference models, rather than (76), allows for closer satisfaction of

$$\int \nabla A dV = \int A \widehat{n} dS \tag{84}$$

even when numerical approximations are used for the derivatives. It is generally thought to be better to use (83) for the pressure gradient, rather than (76). But (83) is not exactly the conservative form because (67) was not a conservative form.

6.4 The divergence $\nabla \cdot \vec{F}$

Recall that in the conformal coordinate system,

$$\nabla \cdot \vec{F} = m^2 \left(\frac{\partial}{\partial X} \frac{F_X}{m} \right)_{Y,Z} + m^2 \left(\frac{\partial}{\partial Y} \frac{F_Y}{m} \right)_{X,Z} + \left(\frac{\partial}{\partial Z} F_Z \right)_{X,Y} . \tag{85}$$

Using steps very similar to the derivation of (83), we find:

$$\nabla\cdot\vec{F} = \frac{\partial \zeta}{\partial Z} \left\{ m^2 \frac{\partial}{\partial X} \left(\frac{F_X}{m} \frac{\partial Z}{\partial \zeta} \right) - \frac{\partial}{\partial \zeta} \left(F_X m \frac{\partial Z}{\partial X} \right) + m^2 \frac{\partial}{\partial Y} \left(\frac{F_Y}{m} \frac{\partial Z}{\partial \zeta} \right) - \frac{\partial}{\partial \zeta} \left(F_Y m \frac{\partial Z}{\partial Y} \right) + \frac{\partial F_Z}{\partial \zeta} \right\} . \tag{86}$$

The above is a conservative form and can be arranged as:

$$\nabla\cdot\vec{F} = \frac{\partial \zeta}{\partial Z} \left\{ m^2 \frac{\partial}{\partial X} \left(\frac{F_X}{m} \frac{\partial Z}{\partial \zeta} \right) + m^2 \frac{\partial}{\partial Y} \left(\frac{F_Y}{m} \frac{\partial Z}{\partial \zeta} \right) + \frac{\partial}{\partial \zeta} \left(F_Z - F_X m \frac{\partial Z}{\partial X} - F_Y m \frac{\partial Z}{\partial Y} \right) \right\} . \tag{87}$$

Let σ be a density of some sort. When $\vec{F}$ is the advective flux of σA, or :

$$\vec{F} = \sigma \vec{U} A, \tag{88}$$

we again find use for $\mathcal{W}$ as defined in (80):

$$\nabla\cdot\left(\sigma\vec{U}A\right) = \frac{\partial \zeta}{\partial Z} \left[m^2 \frac{\partial}{\partial X} \left(\frac{\partial Z}{\partial \zeta} \frac{\sigma U A}{m} \right) + m^2 \frac{\partial}{\partial Y} \left(\frac{\partial Z}{\partial \zeta} \frac{\sigma V A}{m} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\partial Z}{\partial \zeta} \sigma \mathcal{W} A \right) \right] . \tag{89}$$

7 Equations like $\frac{\partial A}{\partial t} + \vec{U} \cdot \nabla A = S_A$

In NWP, we certainly see a lot of forms like

$$\frac{\partial A}{\partial t} + \vec{U} \cdot \nabla A = S_A, \tag{90}$$

where S_A is a general souce term for A. If we retain the advective form of this equation, then, in terrain-following coordinates, it can be written as:

$$\frac{\partial A}{\partial t} + U m \frac{\partial A}{\partial X} + V m \frac{\partial A}{\partial Y} + \mathcal{W} \frac{\partial A}{\partial \zeta} = S_A, \tag{91}$$

where we have used (81). We can also convert the advective form to flux form, if we have an equation like:

$$\frac{\partial \sigma}{\partial t} = \nabla \cdot \left(\sigma \vec{U} \right) . \tag{92}$$

Commonly, we might expect $\sigma = \rho$, where ρ is the familiar mass density. But we need not make an assumption about what σ is; (90) becomes:

$$\frac{\partial}{\partial t} (\sigma A) + \nabla \cdot \left(\sigma \vec{U} A \right) = \sigma S_A, \tag{93}$$

or, using (89),

$$\frac{\partial}{\partial t}(\sigma A)+\frac{\partial \zeta}{\partial Z}\left[m^2\frac{\partial}{\partial X}\left(\frac{\partial Z}{\partial \zeta}\frac{\sigma U A}{m}\right)+m^2\frac{\partial}{\partial Y}\left(\frac{\partial Z}{\partial \zeta}\frac{\sigma V A}{m}\right)+\frac{\partial}{\partial \zeta}\left(\frac{\partial Z}{\partial \zeta}\sigma \mathcal{W} A\right)\right]=\sigma S_A. \tag{94}$$

We define:

$$\sigma^* \equiv \frac{\partial Z}{\partial \zeta}\frac{\sigma}{m}. \tag{95}$$

Now with m and $\frac{\partial Z}{\partial \zeta}$ independent of t, and with m independent of ζ, and can write (94) as

$$\frac{\partial}{\partial t}(\sigma^* A)+m\frac{\partial}{\partial X}(\sigma^* U A)+m\frac{\partial}{\partial Y}(\sigma^* V A)+\frac{\partial}{\partial \zeta}(\sigma^* \mathcal{W} A)=\sigma^* S_A. \tag{96}$$

The pressure equation in ARPS and COAMPS nearly enforces $\nabla\cdot\left(\sigma\vec{U}\right)=0$ with $\sigma=\overline{\rho}\overline{\theta}$, except for the effect of transient sound waves. Here θ is potential temperature and the overbar indicates a time-invariant state. In that case, $\frac{\partial \sigma^*}{\partial t}=0$, and (93) could have been written:

$$\sigma^*\frac{\partial}{\partial t}A+m\frac{\partial}{\partial X}(\sigma^* U A)+m\frac{\partial}{\partial Y}(\sigma^* V A)+\frac{\partial}{\partial \zeta}(\sigma^* \mathcal{W} A)=\sigma^* S_A \tag{97}$$

as a very usable approximation.

It seem to have been once believed, in the construction of ARPS, that ARPS would enforce $\nabla\cdot\left(\sigma\vec{U}\right)=0$ with $\sigma=\overline{\rho}$, rather than $\sigma=\overline{\rho}\overline{\theta}$. Latter versions of ARPS seem to have discovered that (97) needed some improvement with $\sigma=\overline{\rho}$. In ARPS we find the flux form written with additional terms on the right hand side:

$$\sigma^*\frac{\partial}{\partial t}A+m\frac{\partial}{\partial X}(\sigma^* U A)+m\frac{\partial}{\partial Y}(\sigma^* V A)+\frac{\partial}{\partial \zeta}(\sigma^* \mathcal{W} A)=\sigma^* S_A.+Am\frac{\partial}{\partial X}(\sigma^* U)+m\frac{\partial}{\partial Y}(\sigma^* V)+\frac{\partial}{\partial \zeta}(\sigma^* \mathcal{W}) \tag{98}$$

This form follows from

$$\sigma\vec{U}\cdot\nabla A=\nabla\cdot A\sigma\vec{U}-A\nabla\cdot\sigma\vec{U}, \tag{99}$$

without any assumption about $\nabla\cdot\left(\sigma\vec{U}\right)=0$. However, the additional terms on the right-hand side of (98) prevent some of the benefits of working with a flux-form equation.

7.1 extension to a moving vertical coordinate

It is possible to construct models with variables represented on grid points, of constant ζ, that move up or down in the vertical direction. This may be desirable in order to better resolve layer clouds, boundary layers tops, etc. In other words, we consider $Z=Z(\zeta,X,Y,t)$. Assume a grid point is moving with a vertical velocity W_g. We write the time rate of change on the gridpoint as:

$$\frac{\partial A}{\partial \tau} = \frac{\partial A}{\partial t} + W_g \frac{\partial \zeta}{\partial Z}\frac{\partial A}{\partial \zeta}. \tag{100}$$

Note that $\tau = t$, but $\frac{\partial A}{\partial \tau} \neq \frac{\partial A}{\partial t}$. A derivative with respect to τ indicates ζ is held consant; a derivative with respect to t indicates Z is held consant. Now with

$$\mathcal{W}_g \equiv \frac{-Um\frac{\partial Z}{\partial X} - Vm\frac{\partial Z}{\partial Y} + W - W_g}{\frac{\partial Z}{\partial \zeta}}, \tag{101}$$

equation (91) can be written:

$$\frac{\partial A}{\partial \tau} + Um\frac{\partial A}{\partial X} + Vm\frac{\partial A}{\partial Y} + \mathcal{W}_g\frac{\partial A}{\partial \zeta} = S_A. \tag{102}$$

Using (100) in (94), we have

$$\frac{\partial}{\partial \tau}(\sigma A) - W_g\frac{\partial \zeta}{\partial Z}\frac{\partial}{\partial \zeta}(\sigma A) + \frac{\partial \zeta}{\partial Z}\left[m^2\frac{\partial}{\partial X}\left(\frac{\partial Z}{\partial \zeta}\frac{\sigma U A}{m}\right) + m^2\frac{\partial}{\partial Y}\left(\frac{\partial Z}{\partial \zeta}\frac{\sigma V A}{m}\right) + \frac{\partial}{\partial \zeta}\left(\frac{\partial Z}{\partial \zeta}\sigma \mathcal{W} A\right)\right] = \sigma S_A \tag{103}$$

or

$$\frac{1}{m}\frac{\partial Z}{\partial \zeta}\frac{\partial}{\partial \tau}(\sigma A) - W_g\frac{1}{m}\frac{\partial}{\partial \zeta}(\sigma A) + m\frac{\partial}{\partial X}(\sigma^* U A) + m\frac{\partial}{\partial Y}(\sigma^* V A) + \frac{\partial}{\partial \zeta}(\sigma^* \mathcal{W} A) = \sigma S_A. \tag{104}$$

Now with $\frac{\partial Z}{\partial \tau} \equiv W_g$, we have

$$\begin{aligned}\frac{1}{m}\frac{\partial Z}{\partial \zeta}\frac{\partial}{\partial \tau}(\sigma A) - W_g\frac{1}{m}\frac{\partial}{\partial \zeta}(\sigma A) &= \frac{\partial}{\partial \tau}\left(\frac{1}{m}\frac{\partial Z}{\partial \zeta}\sigma A\right) - \sigma A\frac{1}{m}\frac{\partial W_g}{\partial \zeta} - W_g\frac{1}{m}\frac{\partial}{\partial \zeta}\\ &= \frac{\partial}{\partial \tau}(\sigma^* A) - \frac{\partial}{\partial \zeta}\left(\sigma^*\frac{\partial \zeta}{\partial Z}W_g\right).\end{aligned} \tag{105}$$

Again we find use for $\mathcal{W}_g$ in the "neatest" flux form:

$$\frac{\partial}{\partial \tau}(\sigma^* A) + m\frac{\partial}{\partial X}(\sigma^* U A) + m\frac{\partial}{\partial Y}(\sigma^* V A) + \frac{\partial}{\partial \zeta}(\sigma^* \mathcal{W}_g A) = \sigma^* S_A. \tag{106}$$

But, in order to conform to the programming style of ARPS, we would probably work with the form:

$$\sigma^*\frac{\partial A}{\partial \tau} + m\frac{\partial}{\partial X}(\sigma^* U A) + m\frac{\partial}{\partial Y}(\sigma^* V A) + \frac{\partial}{\partial \zeta}(\sigma^* \mathcal{W}_g A) = \sigma^* S_A - A\frac{\partial \sigma^*}{\partial \tau}. \tag{107}$$

For the case $\frac{\partial \sigma}{\partial t} = 0$, we have:

$$\frac{\partial \sigma^*}{\partial \tau} = \frac{1}{m}\sigma \frac{\partial}{\partial \tau}\frac{\partial Z}{\partial \zeta} + \frac{1}{m}\frac{\partial Z}{\partial \zeta} W_g \frac{\partial \zeta}{\partial Z}\frac{\partial \sigma}{\partial \zeta} \tag{108}$$
$$= \frac{1}{m}\sigma \frac{\partial W_g}{\partial \zeta} + \frac{1}{m} W_g \frac{\partial \sigma}{\partial \zeta}$$
$$= \frac{\partial}{\partial \zeta}\left(\frac{\sigma}{m} W_g\right)$$
$$= \frac{\partial}{\partial \zeta}\left(\sigma^* \frac{\partial \zeta}{\partial Z} W_g\right).$$

So, when $\nabla \cdot \left(\sigma \vec{U}\right) = 0$ is a good approximation, an acceptable flux form follows from using (108) in (107):

$$\sigma^* \frac{\partial A}{\partial \tau} + m\frac{\partial}{\partial X}\left(\sigma^* U A\right) + m\frac{\partial}{\partial Y}\left(\sigma^* V A\right) + \frac{\partial}{\partial \zeta}\left(\sigma^* \mathcal{W}_g A\right) = \sigma^* S_A - A\frac{\partial}{\partial \zeta}\left(\sigma^* \frac{\partial \zeta}{\partial Z} W_g\right) \tag{109}$$

If we are working with the (98), appropriate for $\nabla \cdot \left(\sigma \vec{U}\right) \neq 0$, we use

$$\sigma^* \frac{\partial}{\partial t} A = \sigma^* \frac{\partial}{\partial \tau} A - \sigma^* W_g \frac{\partial \zeta}{\partial Z}\frac{\partial A}{\partial \zeta} \tag{110}$$
$$= \sigma^* \frac{\partial}{\partial \tau} A - \frac{\partial}{\partial \zeta}\left(\sigma^* W_g \frac{\partial \zeta}{\partial Z} A\right) + A\frac{\partial}{\partial \zeta}\left(\sigma^* W_g \frac{\partial \zeta}{\partial Z}\right)$$

Now the "extra terms" that have been coded in the fixed-grid version of ARPS can be put to good use. The moving grid form of (98) is:

$$\sigma^* \frac{\partial A}{\partial \tau} + m\frac{\partial}{\partial X}\left(\sigma^* U A\right) + m\frac{\partial}{\partial Y}\left(\sigma^* V A\right) + \frac{\partial}{\partial \zeta}\left(\sigma^* \mathcal{W}_g A\right) = \sigma^* S_A + A\left[m\frac{\partial}{\partial X}\left(\sigma^* U\right) + m\frac{\partial}{\partial Y}\left(\sigma^* V\right) + \frac{\partial}{\partial \zeta}\left(\sigma^* \mathcal{W}_g\right)\right] \tag{111}$$

The finite-difference implementation of either (111) or (102) requires very little new code beyond that of the fixed-grid counterparts, (98) or (91), other than $\mathcal{W} \rightarrow \mathcal{W}_g$.